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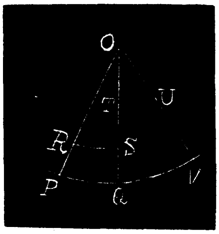
gravity of the man when he is crouching, $ST=b$, $\angle QOP=\alpha$, and $\angle QOV=\theta_0$.

Then the velocity, v, of the man at the point $Q,=\sqrt{2gl(l-k)(1-\cos\alpha)}$, *Bowser's Analytical Mechanics*, page 350, Article 192. Hence, his energy due to his weight $=\dfrac{W}{2g}v^2=W(l-k)(1-\cos\alpha)$. This energy will carry him to V and equals $W(l-k-b)(1-\cos\theta_0)$, since the man rises at the point Q. Since he crouches at the point V, his energy at the point Q on his return is $W(l-k)(1-\cos\theta_0)$. This energy will carry him to a point to the left of S, and the energy expended will be $W(l-k-b)(1-\cos\theta_1)$, where θ_1 is the angle between the vertical and the swing.



According to the principle of the conservation of energy, we have,

$$W(l-k)(1-\cos\alpha)=W(l-k-b)(1-\cos\theta_1)\ldots\ldots\ldots\ldots\ldots(1),$$

$$W(l-k)(1-\cos\theta_0)=W(l-k-b)(1-\cos\theta_2)\ldots\ldots\ldots\ldots\ldots(2),$$

$$W(l--k)(1-\cos\theta_1)=W(l-k-b)(1-\cos\theta_3)\ldots\ldots\ldots\ldots\ldots(3),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$W(l-k)(1-\cos\theta_{2n-1})=W(l-k-b)(1-\cos\theta_{2n-1})\ldots\ldots(2n).$$

Multiplying these equations together member for member, and solving for $1-\cos\theta_{2n}$, we have,

$$1-\cos\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha),\text{ or }\sin^2(\tfrac{1}{2}\theta_{2n})=\left(\frac{l-k}{l-k-b}\right)^{2n}(\sin^2\tfrac{1}{2}\alpha).$$

Whence, $\sin\frac{1}{2}\theta_{2n}=\left(\dfrac{l-k}{l-k-b}\right)^n\sin\frac{1}{2}\alpha$, or $\theta_{2n}=2\sin^{-1}\left[\left(\dfrac{l-k}{l-k-b}\right)\sin\frac{1}{2}\alpha\right]$.

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DIOPHANTINE ANALYSIS.

Conducted by **J. M. COLAW**, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, Chicago, Illinois.

Required all the parallelograms whose sides a, b and diagonals c, d, are rational.

II. **Solution by W. F. KING, Ottowa, Canada.**

The conditions are, $2(a^2+b^2)=c^2+d^2$, with the condition that a, b, c and a, b, d shall be capable of forming triangles (sum of any two sides greater than the third side). That is, if we suppose $a>b$, $c>d$, $a+b$ must be greater than c, and $a-b<d$. These two conditions again are the same, for if $a+b>c$ and $2(a^2+b^2)=c^2+d^2$, $2(a^2+b^2)-(a+b)^2<d^2$ or $a-b<d$.

Let us suppose the numbers involved to be integers. We have $c^2+d^2=2(a^2+b^2)=(a+b)^2+(a-b)^2$. If $c=a+b$, $d=a-b$, the parallelogram vanishes, the angle opposite to the diagonal c becoming 180°. But if not, we have a number c^2+d^2 which is resolvable into the sum of two squares in another way. Hence, as may easily be proved, a^2+b^2 is resolvable into factors, which, by a theorem in the Theory of Numbers, are (whether prime or composite) each expressible as the sum of two squares. Also every prime number of the form $4n+1$ is expressible as the sum of two squares, and every number which is the sum of two squares is the product of prime factors of the form $4n+1$. (The only even prime $2=1^2+1^2$ or a power thereof may also be a factor, which case will be considered further on).

Hence we have a rule to find a and b so as to make c and d rational.

Form a^2+b^2 by multiplying together two or more of the various prime numbers of the form $4n+1$, such as 5, 13, 17, 29, etc.

The product may be expressed in two ways at least as the sum of two squares. Thus we shall have $f^2+g^2=h^2+k^2$.

$\therefore$ $2(f^2+g^2)=(h^2+k)^2+(h-k)^2$, which gives a solution by putting $f=a$, $g=b$, $h+k=c$, $h-k=d$, provided that, (following the condition for a possible triangle) $f-g<h-k$.

If $f-g>h-k$, we must take h and k for a and b ; $f+g$ and $f-g$ for c and d. Then $2(h^2+k^2)=(f+g)^2+(f-g)^2$.

That is, of the two equal sums into which the product has been resolved, take that for a^2+b^2 which has the less difference between its components.

For example, multiply 5 by 13=65.

$65=8^2+1^2=7^2+4^2$ and $7-4=3<8-1$. Hence $a=7$, $b=4$, $h=8$, $k=1$, and $2(7^2+4^2)=(8+1)^2+(8-1)^2=9^2+7^2$.

$\therefore$ $c=9$, $d=7$. And 7, 4, 9 ; 7, 4, 7 are possible sides for triangles. The components of the product can readily be found from the components of the prime factors, thus:

Let $N=(p^2+q^2)(r^2+s^2)=p^2r^2+2pqrs+q^2s^2+q^2r^2-2pqrs+p^2s^2=p^2r^2-2pqrs+q^2s^2+q^2r^2+2pqrs+p^2s^2=(pr+qs)^2(qr-ps)^2=(pr-qs)^2+(qr+ps)^2$.

For example, to resolve 65 into the sum of two squares.

$65=13\times5=(3^2+2^2)(2^2+1^2)$. Here $pr+qs=3\times2+2\times1=8$.

$qr-ps=2\times2-3\times1=1$. $pr-qs=3\times2-2\times1=4$,

$qr+ps=2\times2+3\times1=7$. $\therefore$ $65=8^2+1^2=4^2+7^2$.

A third factor, $r_1{}^2+s_1{}^2$, can be introduced by putting $pr+qs=p_1$, $qr-ps=q_1$, and multiplying out $(p_1{}^2+q_1{}^2)(r_1{}^2+s_1{}^2)$ as before.

Observe that this gives two forms for the product, and two more would be got by putting $pr-qs=p_1$, $qr+ps=q_1$, so that with three factors there will be four forms for the product. These forms may be taken any two together giving $4.3/1.2=6$ solutions for c and d.

In the preceding it has been assumed that a and b have no common factor. If they have one (which may be any number whatever) the preceding investigation will still hold. But such factors of the common measure of a and b as are not primes of the form $4n+1$ will re-appear as common factors of c and d. It is to be noted that if $a^2+b^2=2\times$ a single prime factor a^2+b^2 can be expressed as the sum of two squares in one way only, viz: $a^2+b^2=2(p^2+q^2)=(p+q)^2+(p-q)^2$. For the factors of a^2+b^2 are p^2+q^2 and r^2+s^2 where $r=s=1$, and in multiplying $(p^2+q^2)(r^2+s^2)$ the two expressions $(qr+ps)^2+(pr-qs)^2$ and $(pr+qs)^2+(qr-ps)^2$ become identical when $r=s=1$, and these two cannot be equated to a third and different sum of two squares without factoring p^2+q^2, which is by supposition a prime. Hence when $\frac{1}{2}(a^2+b^2)$ is a prime the solution fails, for we get $2(a^2+b^2)=4(p^2+q^2)=(2p)^2+(2q)^2=(a+b)^2+(a-b)^2$, which does not give a parallelogram. So also when a^2+b^2 is a product of a prime by any odd power of 2. An even power of 2 may however be used, for example,

$$a^2+b^2=260=2^2\times5\times13=2^2(3^2+2^2)(2^2+1^2)=(6^2+4^2)(2^2+1^2)=16^2+2^2$$
$$=14^2+8^2 \text{ and } 2(a^2+b^2)=2(14^2+8^2)=2(16^2+2^2)=18^2+14^2=c^2+d^2.$$

The above discussion made on the assumption that a, b, c, d are integers, is readily extended to give solutions in rational fractions.

Thus $1885=5\times13\times29=42^2+11^2=34^2+27^2$.

$\therefore\ 7^2+(\frac{11}{6})^2=(\frac{17}{3})^2+(\frac{9}{2})^2$ and $2\{(\frac{17}{3})^2+(\frac{9}{2})^2\}=(\frac{53}{6})^2+(\frac{31}{6})^2$.

Note on solution of problem 37, page 151. The failure to give the least values in my solution was due to solving $x_1{}^2-40y_1{}^2=1$; by continued fractions, we obtain positive integral values for x and y, but y does not enter into the required values directly; hence may be fractional. This point was overlooked.

To obtain all these values, let $x=(x_2)\div(z_2)$, $y=(y_2)\div(z_2)$, and then $x_2{}^2-z_2{}^2=40y_2{}^2=10y_3{}^2$. $(x_2+z_2)(x_2-z_2)=10y_3{}^2$. Let $y_3{}^2=p^2q^2$ and $10=$any two factors.

$$\left.\begin{array}{l} x_2+z_2=p^2 \text{ or } 2p^2 \\ x_2-z_2=10q^2 \text{ or } 5q^2 \end{array}\right\}$$

Add and subtract, then

$$\left.\begin{array}{l} x_2=p^2+10q^2 \text{ or } 2p^2+5q^2 \\ z_2=\mp p^2\pm10q^2 \text{ or } \mp2p^2\pm5q^2 \\ y_3=2pq \end{array}\right\}$$

p and q taken at pleasure will give an infinite number of values, integral and fractional. Mr. Gruber's list is correct.
A. H. Bell.